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# On the almost-sure stability condition for a co-dimension two-bifurcation system under the parametric excitation of a real noise

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**Abstract**

In this paper, the almost-sure stability condition for a co-dimension two-bifurcation system on a three-dimensional central manifold, which is parametrically excited by a real noise, is investigated. A model of enhanced generality is developed by assuming the real noise as the first component of an output of a linear filter system—a zero-mean stationary Gaussian diffusion vectoral process, which conforms to the detailed balance condition. The strong mixing condition, which is the essential theoretic basis for the stochastic averaging method, is removed in the present study. To solve the complicated problem encountered in this work, the asymptotic analysis approach and the eigenfunction expansion of the solutions to the relevant Fokker–Planck equations are employed in the construction of the asymptotic expansions of the invariant measures and the maximal Lyapunov exponents for the relevant system.


## 1. Introduction

Recently, the investigation of the almost-sure stability or the maximal Lyapunov exponent for a non-linear stochastic system has emerged as one primary focus of research interests in the field of random dynamical systems as well as stochastic bifurcation. This is mainly attributed to the fact that Lyapunov exponent characterizes the exponential rate of change of the response of a random

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system, and therefore, the sample or the almost-sure stability of the stationary solution of a random dynamical problem depends on the sign of the maximal Lyapunov exponent.

A general method for exact evaluation of the maximal Lyapunov exponent of a linear Ito stochastic differential equation was first presented by Khasminskii [1]. This method was then successfully employed by Kozin and Prodromou [2], Mitchell and Kozin [3], Nishioka [4], and Ariaratnam and Xie [5] to a two-dimensional Ito system.

Among the researches reported to date, only a few results concerned the case of ergodic and colored noise processes. In the work of Arnold et al. [6], and Arnold [7], a procedure for asymptotic analysis was presented and employed to construct the asymptotic expansion of the largest Lyapunov exponent of a two-dimensional system with a real noise excitation. To keep the solution tractable, the infinitesimal generator associated with the noise process was assumed to be a self-adjoint elliptic diffusion operator with an isolated simple zero eigenvalue.

Utilizing the method of stochastic averaging, the asymptotic expansions for Lyapunov exponents for two coupled oscillators with a real noise were obtained by Ariaratnam and Xie [8]. This method was extended by Namachchivaya and Talwar [9] to study the three- and four-dimensional systems under small real noise excitations. Furthermore, the same system was also examined by Namachchivaya and Van Roessel [10]. Instead of using the stochastic averaging method, the perturbation approach proposed by Arnold was applied to construct the asymptotic expansions for the maximal exponents.

For a Van der Pol–Duffing oscillator and a co-dimension two-bifurcation system which possesses one zero eigenvalue and a pair of pure imaginary eigenvalues, that are excited parametrically by a real noise with small intensity which is assumed to be the first component of an output of a linear filter system and conforms to the detailed balance condition [11], Liu and Liew [12,13] obtained the asymptotical expansions of the top Lyapunov exponents for the relevant systems. In these works [12,13], a model of enhanced generality is considered, in which the strong mixing condition which is the essential theoretic basis for the stochastic averaging method was removed. To tackle the complexity encountered in the research process, the asymptotic analysis approach purposed by Arnold et al. [6] and the spectrum representation of the Fokker–Planck (FPK) operator of the linear filter system [11,14,15] are employed in the construction of the asymptotic expansions of the stationary probability density functions and the top Lyapunov exponents for the relevant systems.

This present study is a further extension of the research work in Ref. [13]. According to Pardoux and Wihstutz [16], the asymptotic expression of the top Lyapunov exponent depends on the form of matrix **B**, which is included in the noise excitation term. In this paper, a general form of matrix **B** is considered. Furthermore, for a special case of matrix **B** that caused the complexity of the singular points of phase diffusion process, we will investigate the phenomena that arise from these singular points and discuss the findings thoroughly.

The present paper is organized as follows. In Section 2, we recall the approaches for determining the eigenfunctions and the corresponding eigenvalues of the relevant FPK operator and its adjoint. Section 3 details the formulation of the problem. In Sections 4 and 5, the asymptotical analysis is applied to obtain the expansion of the stationary probabilistic density function. The top Lyapunov exponents for two cases, in which the singularity of the diffusion coefficient arises, are evaluated in Section 6. The conclusion is drawn in Section 7.

## 2. Eigenfunctions and eigenvalues of FPK operator for a linear filter system

Consider a general linear filter system, which is governed by the following stochastic differential system:

$$\dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t) + \dot{\mathbf{W}}(t), \tag{1}$$

where $\mathbf{A} = (a_{ij})_{n\times n}$; $a_{ij}$ are real or complex numbers. $\dot{\mathbf{W}}(t)$ is an $n$-dimensional zero-mean Gaussian white noise with $E(\dot{\mathbf{W}}(t+\tau)\dot{\mathbf{W}}(t)) = \mathbf{V}\delta(\tau)$, $\mathbf{V} = (v_{ij})_{n\times n}$ is a symmetric, non-negative defined constant matrix, and $\mathbf{u} = (u_1, u_2, \ldots, u_n)^{\mathrm{T}}$ is a zero-mean stationary Gaussian diffusion process. In this paper, the matrix $\mathbf{A}$ is assumed to have a complete set of eigenvalues $\alpha_1, \ldots, \alpha_n$ along with the corresponding eigenvectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$, which means that $\alpha_i \neq \alpha_j$ $(i \neq j)$. Furthermore, each eigenvalue $\alpha_i$ is assumed to possess a negative real part, i.e., $R(\alpha_i) < 0$ $(i = 1, 2, \ldots, n)$.

Based on these assumptions, one will find that the probability density function of $\mathbf{u}(t)$ is

$$p_s(\mathbf{u}) = N \exp[-\tfrac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{K}_{\mathbf{u}}^{-1}\mathbf{u}], \quad N = (2\pi)^{-n/2}[\det \mathbf{K}_{\mathbf{u}}]^{1/2}, \tag{2}$$

where $N$ is the normalization constant, and $\mathbf{K}_{\mathbf{u}} = \langle \mathbf{u}(t)\mathbf{u}(t)^{\mathrm{T}} \rangle$ is the covariance matrix. Let $\mathbf{U} = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ be the relevant eigenmatrix of $\mathbf{A}$, one has $\mathbf{D} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \mathrm{diag}[\alpha_1, \alpha_2, \ldots, \alpha_n]$.

For the Markov process $\mathbf{u}(t)$, the differential generator $L_{\mathbf{u}}^*$ and the FPK operator $L_{\mathbf{u}}$ are, respectively, given by

$$L_{\mathbf{u}}^* = a_{ij}u_j \frac{\partial}{\partial u_i} + \frac{1}{2} v_{ij} \frac{\partial^2}{\partial u_i \partial u_j}, \quad L_{\mathbf{u}} = -\frac{\partial}{\partial u_i}[a_{ij}u_j] + \frac{1}{2} v_{ij} \frac{\partial^2}{\partial u_i \partial u_j}, \tag{3}$$

where the repeated indices indicate usual summation. Eigenvalue problems corresponding to the two operators arise as

$$L_{\mathbf{u}}\psi_\lambda(\mathbf{u}) = \lambda\psi_\lambda(\mathbf{u}), \quad L_{\mathbf{u}}^*\psi_{\lambda'}^*(\mathbf{u}) = \lambda'\psi_{\lambda'}^*(\mathbf{u}). \tag{4}$$

It can be verified that $L_{\mathbf{u}}$ and $L_{\mathbf{u}}^*$ possess the same set of eigenvalues [14,15].

Under such a condition, the detailed balance condition [11] is equivalent to

$$p(\mathbf{u}', \tau \mid \mathbf{u}, 0)p_s(\mathbf{u}) = p(\varepsilon\mathbf{u}, \tau \mid \varepsilon\mathbf{u}', 0)p_s(\mathbf{u}'), \quad p_s(\mathbf{u}) = p_s(\varepsilon\mathbf{u}), \tag{5}$$

where $\varepsilon = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, $\varepsilon_i = 1(\varepsilon_i = -1)$ for an even (odd) variable $u_i$. Corresponding to the same eigenvalue $\lambda$, $\psi^*(\mathbf{u})$ and $\psi(\mathbf{u}) = p_s(\mathbf{u})\psi^*(\varepsilon\mathbf{u})$ are, respectively, the eigenfunctions of $L_{\mathbf{u}}^*$ and of $L_{\mathbf{u}}$. Thus, one can investigate only one eigenvalue problem in Eq. (4).

It has been shown by James and Scott [14] and Roy [15] that the solution to Eq. (4) can be chosen as two sets of Hermite polynomials, and corresponding to the eigenvalue $\lambda_{\mathbf{m}} = m_1\alpha_1 + \cdots + m_n\alpha_n$, the eigenfunction of $L_{\mathbf{u}}$ is

$$\psi_{\mathbf{m}}(\varepsilon\mathbf{u}) = \psi_0(\mathbf{u})\psi_{\mathbf{m}}^*(\mathbf{u}) = (-1)^m \frac{\partial^m \psi_0(\mathbf{u})}{\partial w_1^{m_1} \ldots \partial w_n^{m_n}}, \tag{6}$$

where $\mathbf{m} = (m_1, m_2, \ldots, m_n)$; $m = m_1 + m_2 + \cdots + m_n$, $m_i$ $(i = 1, 2, \ldots, n)$ are non-negative integers. For $m = 0$, corresponding to the eigenvalue $\lambda = 0$, $\psi_0(\mathbf{u})$ is equal to the stationary probability density function $p_s(\mathbf{u})$, i.e.,

$$\psi_0(\mathbf{u}) = N \exp[-\tfrac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{K}_{\mathbf{u}}^{-1}\mathbf{u}] = N \exp[-\tfrac{1}{2}\mathbf{v}^{\mathrm{T}}\mathbf{K}_{\mathbf{v}}^{-1}\mathbf{v}] = N \exp[-\tfrac{1}{2}\mathbf{w}^{\mathrm{T}}\mathbf{K}_{\mathbf{v}}\mathbf{w}], \tag{7}$$

where $\mathbf{v} = \mathbf{U}^{-1}\mathbf{u}$, $\mathbf{w} = \mathbf{C}\mathbf{v}$, $\mathbf{C} = [\mathbf{U}^T\mathbf{K}_\mathbf{u}^{-1}\mathbf{U}]$ is a symmetric, positive defined matrix. $\mathbf{K}_v = \mathbf{C}^{-1}$ is the covariance matrix of the stochastic process $\mathbf{v} = \mathbf{U}^{-1}\mathbf{u}$. When $m = 1$, then

$$[\psi_1(\varepsilon\mathbf{u}), \psi_2(\varepsilon\mathbf{u}), \ldots, \psi_n(\varepsilon\mathbf{u})]^T = -\nabla_w\{\exp[-\tfrac{1}{2}\mathbf{w}^T\mathbf{K}_\mathbf{v}\mathbf{w}]\} = \mathbf{U}^{-1}\mathbf{u}\psi_0(\mathbf{u}), \tag{8}$$

where $\psi_k(\mathbf{u})$ $(k = 1, 2, \ldots, n)$ are the eigenfunctions corresponding to the eigenvalues $\alpha_k$, respectively.

The transition probability density of the process $\mathbf{u}(t)$ can be written as

$$p(\mathbf{u}, \tau \mid \mathbf{u}') = \sum_{m_1=0,\ldots,m_n=0}^{\infty} c_\mathbf{m} \exp[\lambda_\mathbf{m}\tau]\psi_\mathbf{m}(\mathbf{u})\psi^*_\mathbf{m}(\mathbf{u}'), \quad \tau \geqslant 0. \tag{9}$$

By combining the initial condition $\lim_{\tau\to 0} p(\mathbf{u}, \tau \mid \mathbf{u}', 0) = \delta(\mathbf{u} - \mathbf{u}')$ and the bi-orthogonality condition [15], each of the coefficients $c_\mathbf{m}$ can be determined as

$$c_\mathbf{m} = \left[\int \mathrm{d}\mathbf{u}\, \psi_\mathbf{m}(\mathbf{u})\psi^*_\mathbf{m}(\mathbf{u})\right]^{-1}. \tag{10}$$

From Eq. (9), one can easily obtain the covariance matrix

$$\mathbf{R}_\mathbf{u}(\tau) = \int \mathrm{d}\mathbf{u} \int \mathrm{d}\mathbf{u}'[\mathbf{u}'\mathbf{u}^T p(\mathbf{u}, \tau \mid \mathbf{u}')p_s(\mathbf{u}')] = \sum_{m_1=0,\ldots,m_n=0}^{\infty} c_\mathbf{m}\mathbf{r}_\mathbf{m}\mathbf{r}_\mathbf{m}^T \exp[\lambda_\mathbf{m}\tau], \tag{11}$$

where vector $\mathbf{r}_\mathbf{m}$ is defined as

$$\mathbf{r}_\mathbf{m} = \int \mathrm{d}\mathbf{u}[\mathbf{u}\psi^*_\mathbf{m}(\mathbf{u})\psi_0(\mathbf{u})] = \int \mathrm{d}\mathbf{u}[\mathbf{u}\psi_\mathbf{m}(\mathbf{u})]. \tag{12}$$

## 3. Formulation

Consider a typical deterministic co-dimension two-bifurcation system which is on a three-dimensional central manifold and possesses one zero-eigenvalue and a pair of pure imaginary eigenvalues [17]:

$$\begin{aligned}
\dot{r} &= \mu_1 r + a_1 rz + (a_2 r^3 + a_3 r^2 z) + O(|r|^4|z|^4),\\
\dot{z} &= \mu_2 z + c_1 r^2 - z^2 + (c_2 r^2 z + c_3 z^3) + O(|r|^4|z|^4),\\
\dot{\Theta} &= \omega + O(|r|^2|z|^2),
\end{aligned} \tag{13}$$

where $\mu_1$ and $\mu_2$ are the unfolding parameters, and $a_1, a_2, a_3, c_1, c_2, c_3$ and $\omega$ are real constants. This normalized form arises in the classic fluid dynamic stability study of Couette flow [17]. In the vicinity of equilibrium point $(r, z, \Theta) = (0, 0, \omega t)$, via the transformation of $r = [x^2 + y^2]^{1/2}$, $\Theta = \arctan[y/x]$, the model of the linearization of the original system (13), which is subjected to a stochastic parametric perturbation, is obtained as

$$\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x} - \varepsilon^2\mathbf{A}_1\mathbf{x} + \varepsilon u_1(t)\mathbf{B}\mathbf{x}, \tag{14}$$

where

$$\mathbf{A}_0 = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \tag{15}$$

and the parameters $\mu_1, \mu_2$ have been rescaled such that

$$\mu_1 = -\varepsilon^2\delta_1, \quad \mu_2 = -\varepsilon^2\delta_2. \tag{16}$$

$u_1(t)$ is the first component of the $\mathbf{u}(t)$, which has been defined in Eq. (1).

The following spherical polar transformation from $(x_1, x_2, x_3)$ to $(\rho, \theta, \phi)$,

$$\begin{aligned} &x_1 = R\cos\theta\sin\phi, \quad x_2 = R\cos\theta\cos\phi, \quad x_3 = R\sin\theta, \\ &\rho = \ln R, \quad \phi(t) = \omega t + \varphi(t), \\ &\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad \phi, \varphi \in [0, 2\pi], \end{aligned} \tag{17}$$

yields a set of equations of the arguments of $\rho, \theta, \phi$ and the noise process $\mathbf{u}$, which was defined in Eq. (1), i.e.,

$$\dot{\rho} = \rho_\varepsilon, \quad \dot{\theta} = \theta_\varepsilon, \quad \dot{\phi} = \phi_\varepsilon, \quad \dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t) + \dot{\mathbf{W}}(t), \tag{18}$$

where

$$\begin{aligned} \rho_\varepsilon &= -\varepsilon^2\rho_2 + \varepsilon u_1(t)\rho_1, \\ \theta_\varepsilon &= -\varepsilon^2\theta_2 + \varepsilon u_1(t)\theta_1, \\ \phi_\varepsilon &= \omega + \varepsilon u_1(t)\phi_1 \end{aligned} \tag{19}$$

and

$$\begin{aligned} \rho_2 &= \delta_1\cos^2\theta + \delta_2\sin^2\theta, \\ \rho_1 &= \tfrac{1}{2}(f_{r2} + f_{z1})\sin 2\theta + f_{r1}\cos^2\theta + f_{z2}\sin^2\theta, \\ \theta_2 &= \tfrac{1}{2}(\delta_2 - \delta_1)\sin 2\theta, \\ \theta_1 &= \tfrac{1}{2}(f_{z2} - f_{r1})\sin 2\theta + (f_{z1}\cos^2\theta - f_{r2}\sin^2\theta), \\ \phi_1 &= f_{\phi 1} + \tan\theta f_{\phi 2}, \\ f_{r1} &= \tfrac{1}{2}[k_1 + k_2\cos 2\phi + k_3\sin 2\phi], \quad f_{r2} = b_{13}\sin\phi + b_{23}\cos\phi, \\ f_{\phi 1} &= \tfrac{1}{2}[k_4 + k_3\cos 2\phi - k_2\sin 2\phi], \quad f_{\phi 2} = b_{13}\cos\phi - b_{23}\sin\phi, \\ f_{z1} &= b_{31}\sin\phi + b_{32}\cos\phi, \quad f_{z2} = b_{33}, \\ k_1 &= b_{22} + b_{11}, \quad k_2 = b_{22} - b_{11}, \quad k_3 = b_{12} + b_{21}, \quad k_4 = b_{12} - b_{21}. \end{aligned} \tag{20}$$

Since the phase processes $\theta$ and $\phi$ are independent of the variable $\rho$, these together with the diffusion process, $\mathbf{u}(t)$, which is defined in Eq. (1), form a vector diffusion process $(\theta(t), \phi(t), \mathbf{u}(t))$

on $[-\pi/2, \pi/2] \times [0, 2\pi] \times R^n$ of dimension $(n+2)$ with the following generator:

$$L_\varepsilon^* = L_0^* + \varepsilon L_1^* + \varepsilon^2 L_2^*,$$
$$L_0^* = L_{\mathbf{u}}^* + \omega \frac{\partial}{\partial \phi}, \quad L_1^* = u_1 \phi_1 \frac{\partial}{\partial \phi} + u_1 \theta_1 \frac{\partial}{\partial \theta}, \quad L_2^* = -\theta_2 \frac{\partial}{\partial \theta}, \tag{21}$$

and the adjoint operator,

$$L_\varepsilon = L_0 + \varepsilon L_1 + \varepsilon^2 L_2,$$
$$L_0 = -\omega \frac{\partial}{\partial \phi} + L_{\mathbf{u}}, \quad L_1 = -u_1 \frac{\partial}{\partial \theta} \theta_1 - u_1 \frac{\partial}{\partial \phi} \phi_1, \quad L_2 = \frac{\partial}{\partial \theta} \theta_2. \tag{22}$$

## 4. Asymptotic analysis

Corresponding to the FPK operator $L_\varepsilon$, the invariant probability density function $p_\varepsilon(\theta, \phi, \mathbf{u})$ satisfies the FPK equation

$$L_\varepsilon p_\varepsilon = (L_0 + \varepsilon L_1 + \varepsilon^2 L_2) p_\varepsilon(\theta, \phi, \mathbf{u}) = 0. \tag{23}$$

In the present paper, $\mathbf{u}(t)$ is assumed to be an ergodic Markov process on $R^n$, and then according to the multiplicative ergodic theorem of Oseledec, the top Lyapunov exponent for system (18) is

$$\lambda_\varepsilon = \langle \rho_\varepsilon, p_\varepsilon \rangle = \int_0^{2\pi} \mathrm{d}\phi \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_{R^n} \rho_\varepsilon(\theta, \phi, \mathbf{u}) p_\varepsilon(\theta, \phi, \mathbf{u}) \, \mathrm{d}\mathbf{u}. \tag{24}$$

For the present work, the assumption $\varepsilon \ll 1$ holds and we do not need the exact solution $p_\varepsilon(\theta, \phi, \mathbf{u})$ of the FPK equation. According to Arnold et al. [6], a formal expansion of

$$p_\varepsilon(\theta, \phi, \mathbf{u}) = p_0(\theta, \phi, \mathbf{u}) + \varepsilon p_1(\theta, \phi, \mathbf{u}) + \cdots + \varepsilon^N p_N(\theta, \phi, \mathbf{u}) + \cdots \tag{25}$$

can be constructed such that

$$L_0 p_0 = 0, \tag{26}$$

$$L_0 p_1 = -L_1 p_0, \tag{27}$$

$$L_0 p_2 = -L_1 p_1 - L_2 p_0, \ldots \tag{28}$$

and hence the top Lyapunov exponent for system (18) may possess an asymptotic expansion as follows:

$$\langle \rho_\varepsilon, p_\varepsilon \rangle = \langle \rho_0, p_0 \rangle + \varepsilon[\langle \rho_1, p_0 \rangle + \langle \rho_0, p_1 \rangle]$$
$$+ \varepsilon^2[\langle \rho_2, p_0 \rangle + \langle \rho_1, p_1 \rangle + \langle \rho_0, p_2 \rangle] + \cdots \tag{29}$$

of which the proof of the validity is needed.

In order to show that Eq. (29) is correct, according to Arnold et al. [6], we construct an adjoint expansion for

$$L_\varepsilon^* F_\varepsilon = \rho_\varepsilon - (f_0 + \varepsilon f_1 + \cdots + \varepsilon^N f_N)$$
$$+ \varepsilon^{N+1}(L_1^* F_N + L_2^* F_{N-1}) + \varepsilon^{N+2}(L_2^* F_N), \tag{30}$$

with

$$F_\varepsilon(\theta,\phi,\mathbf{u}) = F_0(\theta,\phi,\mathbf{u}) + \varepsilon F_1(\theta,\phi,\mathbf{u}) + \cdots + \varepsilon^N F_N(\theta,\phi,\mathbf{u}), \tag{31}$$

where $f_0, f_1, \ldots, f_N$ are functions that do not depend on the variable $\theta$ and $\phi$, but only on $\mathbf{u}(t) \in R^n$, which are chosen such that the sequence of the following problems obtained from Eq. (30),

$$\begin{aligned}
&L_0^* F_0 = \rho_0 - f_0, \quad L_0^* F_1 = \rho_1 - f_1 - L_1^* F_0,\\
&L_0^* F_2 = \rho_2 - f_2 - L_2^* F_0 - L_1^* F_1,\\
&L_0^* F_3 = -f_3 - L_2^* F_1 - L_1^* F_2,\\
&\cdots\\
&L_0^* F_N + L_1^* F_{N-1} + L_2^* F_{N-2} = -f_N
\end{aligned} \tag{32}$$

is solvable.

For the fixed $N$, we introduce $p_\varepsilon^{(N)} = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots + \varepsilon^N p_N$, as the truncated density function of $p_\varepsilon$ and $f_\varepsilon^{(N)} = (f_0 + \varepsilon f_1 + \cdots + \varepsilon^N f_N)$. As $\varepsilon \to 0$, it is easy to verify that the difference between $p_\varepsilon$ and $p_\varepsilon^{(N)}$ is of the order of $\varepsilon^{N+1}$, which is expressed as $\varepsilon^{N+1}(\delta_{\mathrm{p}})$.

With the foregoing preparation, one can arrive at the following equation:

$$\begin{aligned}
&\langle \rho_\varepsilon, p_\varepsilon \rangle - \langle \rho_\varepsilon, p_\varepsilon \rangle_N\\
&\quad = -\varepsilon^{N+1}\{\langle L_1^* F_N + L_2^* F_{N-1}, p_\varepsilon \rangle - \langle L_1^* F_N + L_2^* F_{N-1}, p_\varepsilon^{(N)} \rangle\\
&\qquad + \langle F_\varepsilon, L_1 p_N + L_2 p_{N-1} \rangle - \langle f_\varepsilon^{(N)}, \delta_p \rangle\\
&\qquad - \langle \rho_1, p_N \rangle - \langle \rho_2, p_{N-1} \rangle\}\\
&\quad - \varepsilon^{N+2}\{\langle L_2^* F_N, p_\varepsilon \rangle + \langle F_\varepsilon, L_2 p_N \rangle\\
&\qquad - \langle L_2^* F_N, p_\varepsilon^{(N)} \rangle - \langle \rho_2, p_N \rangle\},
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
&\langle \rho_\varepsilon, p_\varepsilon \rangle_N\\
&\quad = \langle \rho_0, p_0 \rangle + \varepsilon[\langle \rho_1, p_0 \rangle + \langle \rho_0, p_1 \rangle]\\
&\qquad + \varepsilon^2[\langle \rho_2, p_0 \rangle + \langle \rho_1, p_1 \rangle + \langle \rho_0, p_2 \rangle]\\
&\qquad + \cdots + \varepsilon^N[\langle \rho_2, p_{N-2} \rangle + \langle \rho_1, p_{N-1} \rangle + \langle \rho_0, p_N \rangle].
\end{aligned} \tag{34}$$

To furnish expression (33), the following relationship is employed:

$$L_\varepsilon p_\varepsilon^{(N)} = \varepsilon^{N+1}[L_2 p_{N-1} + L_1 p_N] + \varepsilon^{N+2} L_2 p_N. \tag{35}$$

In addition, in the present paper, $\rho_0 = 0$. According to Theorem 3.1 in Section 3 of Arnold et al. [6], suppose $N \geqslant 0$ is fixed, $p_0, p_1, p_2, \ldots, p_N$ and $F_1, F_2, \ldots, F_N$ are such that the inner products on the right side of Eq. (33) are well defined, and

$$\sup_{\phi,\mathbf{u}} |L_1^* F_N + L_2^* F_{N-1}| \leqslant C_1 < \infty, \quad \sup_{\phi,\mathbf{u}} |L_2^* F_N| \leqslant C_2 < \infty. \tag{36}$$

Then the asymptotic expansion (29) for the top Lyapunov exponent of system (18) is available.

In Eq. (25), all the functions $p_\varepsilon(\theta,\phi,\mathbf{u}), p_0(\theta,\phi,\mathbf{u}), \ldots$ are required to be $2\pi$-periodic in variable $\phi$, i.e.,

$$\begin{aligned} p_\varepsilon(\theta,\phi,\mathbf{u}) &= p_\varepsilon(\theta,\phi+2\pi,\mathbf{u}),\\ p_0(\theta,\phi,\mathbf{u}) &= p_0(\theta,\phi+2\pi,\mathbf{u}),\\ p_1(\theta,\phi,\mathbf{u}) &= p_1(\theta,\phi+2\pi,\mathbf{u}), \ldots\ . \end{aligned} \tag{37}$$

The normalization condition of the probabilistic density function $p_\varepsilon(\theta,\phi,\mathbf{u})$ then yields

$$\begin{aligned} &\int_0^{2\pi} \mathrm{d}\phi \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int \mathrm{d}\mathbf{u}\, p_0(\theta,\phi,\mathbf{u}) = 1,\\ &\int_0^{2\pi} \mathrm{d}\phi \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int \mathrm{d}\mathbf{u}\, p_1(\theta,\phi,\mathbf{u}) = \int_0^{2\pi} \mathrm{d}\phi \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int \mathrm{d}\mathbf{u}\, p_2(\theta,\phi,\mathbf{u}) = 0, \ldots\ . \end{aligned} \tag{38}$$

In general, each equation with the form $L_{0p} = q$ must satisfy the following solvability condition, i.e.,

$$\langle q, q^* \rangle = 0, \quad \forall q^* \in Ker\, L_0^* = \{q^* \mid L_0^* q^* = 0\}, \tag{39}$$

where $\langle \cdot,\cdot \rangle$ means the general scalar product, which is defined in Eq. (24), and $L_0^*$ is the adjoint operator of $L_0$ which is defined by

$$L_0^* = \omega \frac{\partial}{\partial \phi} + L_{\mathbf{u}}^*, \tag{40}$$

with $L_{\mathbf{u}}^*$ being the adjoint operator of $L_{\mathbf{u}}$. Then via the scalar product $\langle \cdot,\cdot \rangle$, the following solvability condition is arrived at:

$$\langle q, q^* \rangle = \int_0^{2\pi} \mathrm{d}\phi \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int \mathrm{d}\mathbf{u}\, q^* q = \int_0^{2\pi} \mathrm{d}\phi \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int \mathrm{d}\mathbf{u}\, p L_0^* q^* = 0, \quad \forall q^* \in Ker(L_0^*). \tag{41}$$

Since the operator $L_0^*$ is the sum of the operators imposed separately on variables $\phi$ and $\mathbf{u}$, then each $q^* \in Ker(L_0^*)$ admits a series expansion in terms of the eigenfunctions $\psi_{\mathbf{m}}^*(\mathbf{u})$ of operator $L_0^*$, i.e.,

$$q^*(\theta,\phi,\mathbf{u}) = \sum_{m_1=0,\ldots,m_n=0}^{\infty} q_{\mathbf{m}}^*(\theta,\phi)\psi_{\mathbf{m}}^*(\mathbf{u}). \tag{42}$$

From the definition of the space $Ker(L_0^*)$, we know

$$\left[\omega \frac{\partial}{\partial \phi} + \lambda_{\mathbf{m}}\right] q_{\mathbf{m}}^* = 0. \tag{43}$$

Since $\omega > 0$ and the real part of each eigenvalue except for $\lambda_0 = 0$, to which the associated eigenfunction $\psi_0^*(\mathbf{u}) = 1$, is less than zero, it is apparent that only one non-zero periodic coefficient exists in Eq. (42), namely

$$q_0^*(\theta,\phi) = q_0^*(\theta). \tag{44}$$

It implies that each element in the space $Ker(L_0^*)$ is an arbitrary integrable function of the variable $\theta$; hence, for the present problem, the solvability condition (39) can be written as

$$\int_0^{2\pi} \mathrm{d}\phi \int \mathrm{d}\mathbf{u}\, q(\theta, \phi, \mathbf{u}) = 0. \tag{45}$$

## 5. Expansion for stationary probability density

To obtain the perturbation solution (25) of FPK equation (23), a study on the recurrence equations (26)–(28) will be conducted in the subsequent context.

### 5.1. FPK equation of order $\varepsilon^0$

For Eq. (26), we assume that the solution $p_0(\theta, \phi, \mathbf{u})$ possess a series expansion in terms of eigenfunctions $\psi_{\mathbf{m}}(\mathbf{u})$ of $L_{\mathbf{u}}$, such that

$$\begin{aligned} p_0(\theta, \phi, \mathbf{u}) &= \sum_{m_1=0,\dots,m_n=0}^{\infty} p_{\mathbf{m}}^{(0)}(\theta, \phi)\psi_{\mathbf{m}}(\mathbf{u}) \\ &= p_0^{(0)}(\theta, \phi)\psi_0(\mathbf{u}) + \sum_{k=1}^{n} p_k^{(0)}(\theta, \phi)\psi_k(\mathbf{u}) + \sum_{k,l=1}^{n} p_{kl}^{(0)}(\theta, \phi)\psi_{kl}(\mathbf{u}) + \cdots . \end{aligned} \tag{46}$$

Then the coefficients $p_{\mathbf{m}}^{(0)}(\theta, \phi)$ are, respectively, the solutions of

$$\left[-\omega \frac{\partial}{\partial \phi} + \lambda_{\mathbf{m}}\right] p_{\mathbf{m}}^{(0)} = 0. \tag{47}$$

There exists only one non-zero periodic solution, $p_0^{(0)}(\theta, \phi) = p_0^{(0)}(\theta)$, which corresponds to the eigenvalue $\lambda_0 = 0$. Employing the normalization condition (38), we found that the solution to Eq. (26) takes the following expression:

$$p_0(\theta, \phi, \mathbf{u}) = \frac{1}{2\pi} F(\theta)\psi_0(\mathbf{u}), \tag{48}$$

where $F(\theta)$ is a function of $\theta$ yet to be determined by the solvability condition for the equation of order $\varepsilon^2$.

### 5.2. FPK equation of order $\varepsilon$

Consider Eq. (27). Substitution of Eq. (48) into the right side of Eq. (27) yields

$$\begin{aligned} L_0 p_1(\theta, \phi, \mathbf{u}) &= \frac{u_1\psi_0(\mathbf{u})}{2\pi}\left[\theta_1 \frac{\partial F(\theta)}{\partial \theta} + \left[\frac{\partial \theta_1}{\partial \theta} + \frac{\partial \phi_1}{\partial \phi}\right] F(\theta)\right] \\ &= \frac{u_1\psi_0(\mathbf{u})}{2\pi}[M_0 + M_1 \cos 2\phi + M_2 \sin 2\phi + M_3 \cos \phi + M_4 \sin \phi], \end{aligned} \tag{49}$$

where

$$M_0 = \frac{1}{2}\left(b_{33} - \frac{k_1}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\theta}[\sin 2\theta F(\theta)],$$
$$M_1 = -k_2\Lambda_1, \quad M_2 = -k_3\Lambda_1, \quad \Lambda_1 = F(\theta) + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}\theta}[\sin 2\theta F(\theta)],$$
$$M_3 = -b_{23}\,\mathrm{tg}\,\theta F(\theta) + \frac{\mathrm{d}}{\mathrm{d}\theta}[(b_{32}\cos^2\theta - b_{23}\sin^2\theta)F(\theta)],$$
$$M_4 = -b_{13}\,\mathrm{tg}\,\theta F(\theta) + \frac{\mathrm{d}}{\mathrm{d}\theta}[(b_{31}\cos^2\theta - b_{13}\sin^2\theta)F(\theta)]. \tag{50}$$

It is known that the function $u_1\psi_0(\mathbf{u})$ can be expressed as a linear combination of eigenfunctions $\{\psi_k(\mathbf{u}),\ k = 1, 2, \ldots, n\}$ of order $m = 1$ [15], such that

$$u_1\psi_0(\mathbf{u}) = \sum_{k=1}^{n} \gamma_k\psi_k(\mathbf{u}). \tag{51}$$

In fact, the coefficient $\gamma_k$ is the first element of the vector $\mathbf{r}_k/c_k$, which is defined in Section 2 for the order $m = 1$.

Expanding $p_1(\theta, \phi, \mathbf{u})$ in terms of the eigenfunctions $\{\psi_{\mathbf{m}}(\mathbf{u})\}$ yields

$$p_1(\theta, \phi, \mathbf{u}) = p_0^{(1)}(\theta, \phi)\psi_0(\mathbf{u}) + \sum_{k=1}^{n} p_k^{(1)}(\theta, \phi)\psi_k(\mathbf{u}) + \sum_{k,l=1}^{n} p_{kl}^{(1)}(\theta, \phi)\psi_{kl}(\mathbf{u}) + \cdots . \tag{52}$$

Substitution of Eqs. (52) and (49) into Eq. (27) leads to the fact that $p_{\mathbf{m}}^{(1)}(\theta, \phi)$ are, respectively, governed by the following equations:

$$\left[-\omega\frac{\partial}{\partial\phi} + \lambda_k\right]p_k^{(1)}(\theta, \phi) = \begin{cases} \dfrac{\gamma_k}{2\pi}\{M_0 + M_1\cos 2\phi + M_2\sin 2\phi + M_3\cos\phi + M_4\sin\phi\}, & m = \sum_{i=1}^{n} m_i = 1, \\ 0, & m = \sum_{i=1}^{n} m_i \neq 1, \end{cases} \tag{53}$$

corresponding to the eigenvalue $\lambda_0 = 0$, the solution is

$$p_0^{(1)}(\theta, \phi) = \frac{1}{2\pi}p_0^{(1)}(\theta), \tag{54}$$

where $p_0^{(1)}(\theta)$ is a function yet to be determined, which satisfies the normalization condition, i.e.,

$$\int_0^{2\pi} p_0^{(1)}(\theta)\,\mathrm{d}\theta = 0. \tag{55}$$

As we know, each $p_{\mathbf{m}}^{(1)}(\theta, \phi)$ $(m \geqslant 2)$ is a periodic function of the variable $\phi$ with period $2\pi$ and $p_1(\theta, \phi, \mathbf{u})$ satisfies the normalization condition (38), therefore, it is clear that for the condition $m \geqslant 2$,

$$p_{\mathbf{m}}^{(1)}(\theta,\phi)=0, \quad m \geqslant 2. \tag{56}$$

For the case of $m=1$ and $\lambda_k=\alpha_k$. By solving Eq. (53) directly, we can obtain the solution functions as

$$\begin{aligned} p_k^{(1)}(\theta,\phi) = \frac{\gamma_k}{2\pi}\Bigg\{ & \frac{1}{\alpha_k}M_0 + \frac{1}{(2\omega)^2+\alpha_k^2}\{[\alpha_k M_1 + 2\omega M_2]\cos 2\phi + [\alpha_k M_2 - 2\omega M_1]\sin 2\phi\} \\ & + \frac{1}{\omega^2+\alpha_k^2}\{[\alpha_k M_3 + \omega M_4]\cos\phi + [\alpha_k M_4 - \omega M_3]\sin\phi\}\Bigg\}. \end{aligned} \tag{57}$$

Finally, by summarizing the foregoing results, we found that $p_1(\theta,\phi,\mathbf{u})$ takes the expression as

$$p_1(\theta,\phi,\mathbf{u}) = \frac{1}{2\pi}p_0^{(1)}(\theta)\psi_0(\mathbf{u}) + \sum_{k=1}^{n} p_k^{(1)}(\theta,\phi)\psi_k(\mathbf{u}), \tag{58}$$

where $p_0^{(1)}(\theta)$ is a function to be determined by the solvability condition of order $\varepsilon^3$. Therefore, by evaluating the asymptotic expansion for the top Lyapunov exponent, we found that $p_0^{(1)}(\theta)$ is not contributing to the expression of the top Lyapunov exponent. In addition, each $p_k^{(1)}(\theta,\phi)$ contains the function $F(\theta)$, which should be determined by the solvability condition of Eq. (28).

### 5.3. Solvability condition and FPK equation

To determine the function $F(\theta)$ in Eqs. (48) and (58), the solvability condition of Eq. (28) will be investigated as follows. Since the solvability condition of Eq. (28) is

$$-\int_0^{2\pi} \mathrm{d}\phi \int \mathrm{d}\mathbf{u}[L_1 p_1 + L_2 p_0] = 0, \tag{59}$$

substitution of Eq. (58) into Eq. (59) yields

$$-\int \mathrm{d}\mathbf{u}\int_0^{2\pi} L_1 p_1\,\mathrm{d}\phi = I_1 I_2 + \sum_{k=1}^{n} I_3^{(k)} I_4^{(k)}, \tag{60}$$

where

$$\begin{aligned} & I_1 = \int \mathrm{d}\mathbf{u}[u_1\psi_0(\mathbf{u})] = \int \mathrm{d}\mathbf{u}[u_1 p_s(\mathbf{u})] = 0, \\ & I_3^{(k)} = \int \mathrm{d}\mathbf{u}[u_1\psi_k(\mathbf{u})] = \xi_k, \quad I_4^{(k)} = \frac{\partial}{\partial\theta}\left[\int_0^{2\pi}\theta_1 p_k^{(1)}\,\mathrm{d}\phi\right], \quad k=1,2,\ldots,n. \end{aligned} \tag{61}$$

Then via computation, we know

$$\begin{aligned}
&-\int_0^{2\pi} \mathrm{d}\mathbf{u} \int L_1 p_1 \, \mathrm{d}\phi \\
&= -\frac{1}{8}\beta_1 \left[\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}[\sin^2(2\theta)F] - \frac{\mathrm{d}}{\mathrm{d}\theta}[\sin(4\theta)F]\right] \\
&- \frac{1}{16}\beta_2 \left[\frac{\mathrm{d}}{\mathrm{d}\theta}[\sin(2\theta)F] + \frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}[\sin^2(2\theta)F] - \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}\theta}[\sin(4\theta)F]\right] \\
&- \frac{3}{8}\kappa_0 \frac{\mathrm{d}}{\mathrm{d}\theta}[\sin(2\theta)F] \\
&- \frac{1}{16}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}[(\kappa_1 \cos^2(2\theta) - 2\kappa_2 \cos(2\theta) + \kappa_3)F] \\
&- \frac{1}{16}\frac{\mathrm{d}}{\mathrm{d}\theta}[(\kappa_1 \sin(4\theta) - 2\kappa_4 \sin(2\theta) + 4\kappa_5 \tan(\theta))F],
\end{aligned} \tag{62}$$

where

$$\begin{aligned}
\beta_1 &= S_{u_1}(0)[b_{33} - \tfrac{1}{2}k_1]^2, \\
\beta_2 &= S_{u_1}(2\omega)[k_2^2 + k_3^2], \\
\kappa_0 &= \Phi_{u_1}(\omega)[-b_{13}b_{32} + b_{23}b_{31}], \\
\kappa_1 &= [(b_{13} + b_{31})^2 + (b_{23} + b_{32})^2]S_{u_1}(\omega), \\
\kappa_2 &= [(b_{13}^2 - b_{31}^2) + (b_{23}^2 - b_{32}^2)]S_{u_1}(\omega), \\
\kappa_3 &= [(b_{13} - b_{31})^2 + (b_{23} - b_{32})^2]S_{u_1}(\omega), \\
\kappa_4 &= [(b_{13} + b_{31})(2b_{13} - b_{31}) + (b_{23} + b_{32})(2b_{23} - b_{32})]S_{u_1}(\omega), \\
\kappa_5 &= [b_{13}^2 + b_{23}^2]S_{u_1}(\omega).
\end{aligned} \tag{63}$$

For the stationary, stochastic vector process $\mathbf{u}(t)$, via the definition of the spectral density function [11], we can obtain the explicit expression of the spectral density function of $u_1(t)$ [15], i.e.,

$$\begin{aligned}
S_{u_1}(\omega) &= 2\int_0^{+\infty} R_u(\tau)\cos(\omega\tau)\,\mathrm{d}\tau = -\sum_{k=1}^{n} \gamma_k \xi_k \frac{2\alpha_k}{\alpha_k^2 + \omega^2}, \\
\Phi_{u_1}(\omega) &= 2\int_0^{+\infty} R_u(\tau)\sin(\omega\tau)\,\mathrm{d}\tau = -\sum_{k=1}^{n} \gamma_k \xi_k \frac{2\omega}{\alpha_k^2 + \omega^2}.
\end{aligned} \tag{64}$$

Evaluation of the second term of the left side of Eq. (51) leads to

$$-\int_0^{2\pi} \mathrm{d}\phi \int \mathrm{d}\mathbf{u}[L_2 p_0] = -\int_0^{2\pi} \mathrm{d}\phi \int \mathrm{d}\mathbf{u}\left[\frac{\psi_0(\mathbf{u})}{2\pi}\frac{\partial[\theta_2 F)]}{\partial\theta}\right] = \frac{\delta_1 - \delta_2}{2}\frac{\mathrm{d}[\sin 2\theta F]}{\mathrm{d}\theta}. \tag{65}$$

Finally, by summarizing the results of Eqs. (62) and (65), we find that the solvability condition (51) is equivalent to the following standard FPK equation:

$$\frac{1}{2}\frac{d^2}{d\theta^2}[\sigma^2(\theta)F(\theta)] - \frac{d}{d\theta}[\mu(\theta)F(\theta)] = 0, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{66}$$

in which the relevant diffusion coefficient and drift coefficient are, respectively,

$$\sigma^2(\theta) = (4\beta_1 + \tfrac{1}{2}\beta_2)\sin^2 2\theta + 2\kappa_1\cos^2 2\theta - 4\kappa_2\cos 2\theta + 2\kappa_3,$$

$$\begin{aligned}\mu(\theta) = &\{[2\beta_1 + \tfrac{1}{4}\beta_2] - \kappa_1\}\sin 4\theta \\ &+ \{8(\delta_1 - \delta_2) + 2\kappa_4 - \beta_2 - 6\kappa_0\}\sin 2\theta - 4\kappa_5\tan\theta.\end{aligned} \tag{67}$$

In order to make the problem tractable, we assume that

$$b_{31} = b_{13}, \quad b_{32} = b_{23}, \tag{68}$$

from which we can obtain that

$$\begin{aligned}&\kappa_0 = \kappa_2 = \kappa_3 = 0, \\ &\kappa_1 = 4\kappa_5, \quad \kappa_4 = 2\kappa_5.\end{aligned} \tag{69}$$

Eq. (67) can then be changed to

$$\sigma^2(\theta) = (4\beta_1 + \tfrac{1}{2}\beta_2)\sin^2 2\theta + 8\kappa_5\cos^2 2\theta,$$

$$\begin{aligned}\mu(\theta) = &\{[2\beta_1 + \tfrac{1}{4}\beta_2] - 4\kappa_5\}\sin 4\theta \\ &+ \{8(\delta_1 - \delta_2) + 4\kappa_5 - \beta_2\}\sin 2\theta - 4\kappa_5\tan\theta.\end{aligned} \tag{70}$$

In view of FPK Eq. (66), the process of $\theta(t)$ can be treated as a diffusion process on interval $[-\pi/2, \pi/2]$ with the relevant drift parameter $\mu(\theta)$ and the diffusion parameter $\sigma^2(\theta)$, respectively. In order to determine the solution for Eq. (66), the diffusion behaviors of such a process at the boundaries of $\theta = \pm\pi/2$ and other singular points within $[-\pi/2, \pi/2]$ should be investigated.

The details of the definition and classification of singular points for one-dimensional diffusion processes can be found in Ref. [18], from which, we know that the first kind of singular points is the one at which $\sigma^2(\theta)$ vanishes, and the second kind is that at which $\mu(\theta)$ goes to infinity. With these definitions, we can conclude that on the interval $[-\pi/2, \pi/2]$, $\theta = \pm\pi/2$ are the singular points of the second kind.

For the singular boundary $x_s$ ($x_l$, $x_r$ represent the left and right boundaries, respectively) of the second kind, the diffusion exponent $\alpha_s$, the drift exponent $\beta_s$ and the character number $c_s$ are introduced [18]:

$$\begin{aligned}&\sigma^2(x) = O(|x - x_s|^{-\alpha_s}), \quad \alpha_s \geqslant 0, \ \ x \to x_s, \\ &\mu(x) = O(|x - x_s|^{-\beta_s}), \quad \beta_s \geqslant 0, \ \ x \to x_s, \\ &c_l = \lim_{x \to x_l^+} \frac{2\mu(x)[x - x_l]^{\beta_l - \alpha_l}}{\sigma^2(x)}, \quad c_r = -\lim_{x \to -x_r^-} \frac{2\mu(x)[x_r - x]^{\beta_r - \alpha_r}}{\sigma^2(x)}.\end{aligned} \tag{71}$$

On $[-\pi/2, \pi/2]$, we consider only the situation of $\theta = \pi/2$. For the case of $\theta = -\pi/2$, the result is similar. While $\theta \to \pi/2$, we obtain

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sin[(\pi/2)-\theta]} \propto \frac{1}{[(\pi/2)-\theta]}. \tag{72}$$

Then Eq. (71) leads to

$$\alpha_{\rm r} = 0, \quad \beta_{\rm r} = 1,$$

$$c_r = -\lim_{\theta\to\pi^-/2} \frac{2\mu(\theta)[(\pi/2)-\theta]^{1-0}}{\sigma^2(\theta)} = -\frac{-8\kappa_5 \sin[(\pi/2)][(\pi/2)-\theta]^{1-0}}{8\kappa_5\cos^2\pi\cos[(\pi/2)]} = 1 \tag{73}$$

which are, respectively, the diffusion, drift exponents and character value at $\theta = \pi/2$. After checking out these results with the terms in Lin and Cai [18, Table 3], which gives a detailed classification of the singular boundaries of the second kind, we know that $\pi/2$ is an entrance of $[-\pi/2, \pi/2]$ and the result is the same as $\theta = -\pi/2$, i.e., $\theta = -\pi/2$ is another entrance.

For the diffusion process $\theta$, its scale and speed densities are defined, respectively, as [19]

$$s(\theta) = \exp[-E(\theta)], \quad m(\theta) = \frac{1}{\sigma^2(\theta)s(\theta)},$$

$$E(\theta) = \int [2\mu(\theta)\sigma^{-2}(\theta)]\,{\rm d}\theta \tag{74}$$

and in addition, the relevant scale and speed measures are

$$S(\theta) = \int^{\theta} s(x)\,{\rm d}x, \quad M(\theta) = \int^{\theta} m(x)\,{\rm d}x. \tag{75}$$

For Eq. (66), its solution can be represented as

$$F(\theta) = m(\theta)[C_1 S(\theta) + C], \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{76}$$

where $C_1$ and $C$ are constants, which will be determined by the normality and boundary conditions.

For the case of $A_1 \neq A_2$ and $A_1 > A_2$, we know

$$\begin{aligned} &E(\theta) = E^{(1)} + E^{(2)} + E^{(3)},\\ &E^{(1)} = \tfrac{1}{2}\ln[1-(1-\tau_1)\cos^2 2\theta],\\ &E^{(2)} = -\frac{B_2}{A_1}\frac{1}{\sqrt{1-\tau_1}}\,{\rm arctanh}[\sqrt{1-\tau_1}\cos 2\theta],\\ &E^{(3)} = \ln\cos\theta - \tfrac{1}{4}\ln[1-(1-\tau_1)\cos^2 2\theta] - \tfrac{1}{2}\sqrt{1-\tau_1}\,{\rm arctanh}[\sqrt{1-\tau_1}\cos 2\theta], \end{aligned} \tag{77}$$

where

$$\begin{aligned} &A_1 = 4\beta_1 + \tfrac{1}{2}\beta_2, \quad A_2 = 8\kappa_5,\\ &B_1 = 2\beta_1 + \tfrac{1}{4}\beta_2 - 4\kappa_5, \quad B_2 = 8[\delta_1 - \delta_2] + 4\kappa_5 - \beta_2, \quad B_3 = -4\kappa_5,\\ &0 < \tau_1 = \frac{A_2}{A_1} < 1. \end{aligned} \tag{78}$$

And for another case $A_1 < A_2$ ($\tau_1 > 1$), we obtain

$$\begin{aligned} E^{(1)} &= \tfrac{1}{2}\ln[1+(\tau_1-1)\cos^2 2\theta], \\ E^{(2)} &= -\frac{B_2}{A_1}\frac{1}{\sqrt{\tau_1-1}}\arctan[\sqrt{\tau_1-1}\cos 2\theta], \\ E^{(3)} &= \ln|\cos\theta| - \tfrac{1}{4}\ln[1+(\tau_1-1)\cos^2 2\theta] + \tfrac{1}{2}\sqrt{\tau_1-1}\arctan[\sqrt{\tau_1-1}\cos 2\theta]. \end{aligned} \tag{79}$$

These expressions lead to

$$m(\theta) = \begin{cases} \dfrac{1}{A_1}\cos\theta \dfrac{[1-\sqrt{1-\tau_1}\cos 2\theta]^{(1/2)\eta_1-(3/4)}}{[1+\sqrt{1-\tau_1}\cos 2\theta]^{(1/2)\eta_1+(3/4)}}, & \tau_1 < 1, \\ \dfrac{1}{A_1}\cos\theta \dfrac{\exp[-\eta_1 \arctan[\sqrt{\tau_1-1}\cos 2\theta]]}{[1+(\tau_1-1)\cos^2 2\theta]^{3/4}}, & \tau_1 > 1, \end{cases} \tag{80}$$

where

$$\eta_1 = \begin{cases} \dfrac{B_1+B_2}{A_1\sqrt{1-\tau_1}}, & \tau_1 < 1, \\ \dfrac{B_1+B_2}{A_1\sqrt{\tau_1-1}}, & \tau_1 > 1. \end{cases} \tag{81}$$

Since the two boundaries are both entrances, we know that in Eq. (76), $C_1 = 0$, and $C$ can be determined by the condition $\int_{-\pi/2}^{\pi/2} F(\theta)\,\mathrm{d}\theta = 1$. Thus, the solution to Eq. (66), the invariant measure, can be expressed as

$$F(\theta) = C\begin{cases} \dfrac{1}{A_1}\cos\theta \dfrac{[1-\sqrt{1-\tau_1}\cos 2\theta]^{(1/2)\eta_1-(3/4)}}{[1+\sqrt{1-\tau_1}\cos 2\theta]^{(1/2)\eta_1+(3/4)}}, & \tau_1 < 1, \\ \dfrac{1}{A_1}\cos\theta \dfrac{\exp[-\eta_1 \arctan[\sqrt{\tau_1-1}\cos 2\theta]]}{[1+(\tau_1-1)\cos^2 2\theta]^{(3/4)}}, & \tau_1 > 1. \end{cases} \tag{82}$$

### 5.4. Two special cases

From Eq. (82), we know that it is impossible to obtain the analytical results for the maximal Lyapunov exponent. Furthermore, according to Pardoux and Wihstutz [16], the expressions of top Lyapunov exponents depend on the forms of the matrix **B**, therefore in this paper, two special cases for the coefficient matrix **B** in Eq. (15) are considered, i.e.,

*Case* I: $A_1 = A_2 = A$, which imply that

$$\begin{aligned} 4S_{u_1}(\omega)[b_{13}^2 + b_{23}^2] = {} & 2S_{u_1}(0)[2b_{33} - b_{11} - b_{22}]^2 \\ & + S_{u_1}(2\omega)[(b_{22}-b_{11})^2 + (b_{12}+b_{21})^2]. \end{aligned} \tag{83}$$

For the case of white noise excitation along with the facts that $b_{33} = b_{22} = b_{11}$, $b_{21} = b_{12}$, of which the coefficient matrix **B** in Eq. (15) is of the following form:

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{11} & b_{23} \\ b_{13} & b_{23} & b_{11} \end{bmatrix}, \tag{84}$$

the condition expression (83) is equivalent to

$$b_{12}^2 = 4[b_{13}^2 + b_{23}^2]. \tag{85}$$

*Case* II: $b_{33} = b_{22} = b_{11}$, $b_{21} = -b_{12}$, from which we deduce that

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ -b_{12} & b_{11} & b_{23} \\ b_{13} & b_{23} & b_{11} \end{bmatrix}, \quad \beta_1 = \beta_2 = 0. \tag{86}$$

In the subsequent procedure, for each case, we will investigate the stationary solution to FPK equation (66).

For the first case which is under the condition that $\sigma^2(\theta) = A$, $B_1 = 0$ and $2B_3 = -A$, we can obtain

$$\begin{aligned} E(\theta) &= \frac{2B_2}{A}\sin^2\theta + \ln|\cos\theta|, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \\ s(\theta) &= |\sec\theta|\exp[-\alpha\sin^2\theta], \\ m(\theta) &= \frac{1}{A}|\cos\theta|\exp[\alpha\sin^2\theta], \\ \alpha &= \frac{2B_2}{A} = \frac{2(\delta_1 - \delta_2)}{\kappa_5} + \left(1 - \frac{\beta_2}{4\kappa_5}\right). \end{aligned} \tag{87}$$

In order to make the problem tractable, we divide the interval $[-\pi/2, \pi/2]$ into two subsets as $[-\pi/2, 0)$ and $(0, \pi/2]$. Since the solution problem on $[-\pi/2, 0]$ is the same as the situation on $[0, \pi/2]$, we will only investigate the solution problem on $[0, \pi/2]$. On both $[-\pi/2, 0)$ and $(0, \pi/2]$, $\theta = 0$ is not a singular point. To investigate the diffusion behavior at $\theta = 0$, we employ the concepts of the scale and speed densities. As

$$s(\theta) = \sec\theta\exp[-\alpha\sin^2\theta], \quad \theta \in \left[0, \frac{\pi}{2}\right], \tag{88}$$

on $(0, \pi/2]$ and in the neighborhood of $\theta = 0$, for the two cases of $\alpha > 0$ and $\alpha \leqslant 0$, respectively,

$$\begin{aligned} &\mathrm{e}^{-\alpha}\sec\theta \leqslant s(\theta) \leqslant \sec\theta, \quad \alpha > 0, \\ &\mathrm{e}^{-\alpha}\sec\theta \geqslant s(\theta) \geqslant \sec\theta, \quad \alpha \leqslant 0. \end{aligned} \tag{89}$$

Then for the scale measure $S(0, \theta]$, the following two inequalities are tenable:

$$\begin{aligned} &-\infty < \int_0^\theta \{[\mathrm{e}^{-\alpha}]\sec x\}\,\mathrm{d}x \leqslant S(0,\theta] \leqslant \int_0^\theta \sec x\,\mathrm{d}x < +\infty, \quad \alpha > 0, \\ &-\infty < \int_0^\theta \sec x\,\mathrm{d}x \leqslant S(0,\theta] \leqslant \int_0^\theta \{[\mathrm{e}^{-\alpha}]\sec x\}\,\mathrm{d}x < +\infty, \quad \alpha \leqslant 0. \end{aligned} \tag{90}$$

For the speed measure $M(0,\theta]$, via the definition expression (75), we obtain

$$M(0,\theta] = \int_0^{\sin\theta} \frac{1}{A}\{\exp[\alpha x^2]\}\,\mathrm{d}x = \frac{1}{A}\,Erfi[\sin\theta], \quad \alpha > 0,$$
$$M(0,\theta] = \int_0^{\sin\theta} \frac{1}{A}\{\exp[\alpha x^2]\}\,\mathrm{d}x = \frac{1}{A}\,Erf[\sin\theta], \quad \alpha \leqslant 0, \tag{91}$$

where *Erfi* and *Erf* are the error functions.

Eqs. (90) and (91) tell us that the two measures are both finite, thus according to the definition of a reflecting boundary [19], we know that $\theta = 0$ is a reflecting boundary for the two intervals, which is shown in Fig. 1. With this result, we can conclude that the diffusion process evolves on $[-\pi/2, 0)$ and $(0, \pi/2]$ separately, and the solution to Eq. (66) will be analyzed on the two intervals, respectively.

To system (66) restricted on $(0, \pi/2]$, the solution is

$$F(\theta) = \begin{cases} \dfrac{2\sqrt{\alpha}}{\sqrt{\pi}\,Erfi[\sqrt{\alpha}]}\cos\theta\exp[\alpha\sin^2\theta], & \alpha > 0, \\ \dfrac{2\sqrt{-\alpha}}{\sqrt{\pi}\,Erf[\sqrt{-\alpha}]}\cos\theta\exp[\alpha\sin^2\theta], & \alpha \leqslant 0, \end{cases} \qquad \theta \in \left[0, \frac{\pi}{2}\right]. \tag{92}$$

We can then verify easily that on $[-\pi/2, 0)$, the stationary probability density is of the same expression.

For the second case, assumption (86) leads to

$$\sigma^2(\theta) = 8\kappa_5\cos^2 2\theta,$$
$$\mu(\theta) = -4\kappa_5\sin 4\theta + \{8(\delta_1 - \delta_2) + 4\kappa_5\}\sin 2\theta - 4\kappa_5\tan\theta. \tag{93}$$

Since at $\theta = \pm\pi/4$, $\sigma^2(\theta) = 0$, $\theta = \pm\pi/4$ are singular points of the first kind, and at points $\theta = \pm\pi/2$, $\mu(\theta) = -\infty$, so $\theta = \pm\pi/2$ are singular points of the second kind. Because of the different diffusion behaviors of the singular points, the interval $[-\pi/2, \pi/2]$ should be divided into

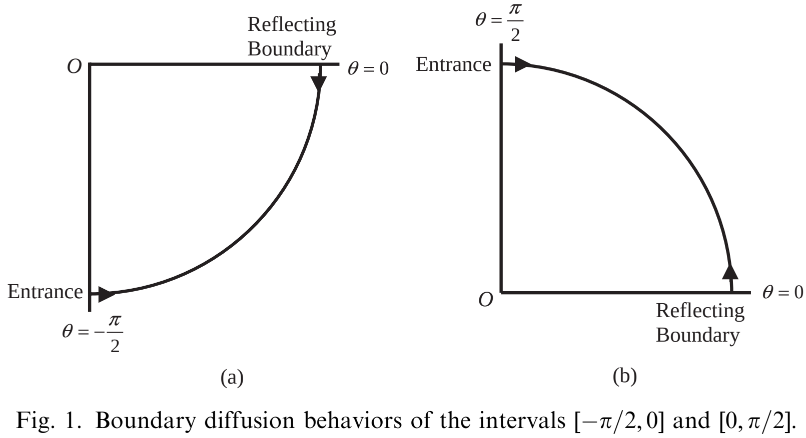


Fig. 1. Boundary diffusion behaviors of the intervals $[-\pi/2, 0]$ and $[0, \pi/2]$.

three sub-intervals, i.e., $[-\pi/2, -\pi/4]$, $[-\pi/4, \pi/4]$ and $[\pi/4, \pi/2]$, of which the solutions to Eq. (66) should be investigated. First, we investigate the diffusion behaviors of the singular points.

On $[-\pi/2, -\pi/4]$, according to the definition expressions in Eq. (71), we know that at the left boundary $\theta = -\pi/2$, the diffusion and the drift exponents and the character value are

$$\alpha_l = 0, \quad \beta_r = 1,$$

$$c_r = \lim_{\theta \to -\pi^+/2} \frac{2\mu(\theta)[\theta + (\pi/2)]^{1-0}}{\sigma^2(\theta)} = \frac{-2 \times 4\kappa_5 \tan\theta[\theta + (\pi/2)]}{8\kappa_5} = 1 \tag{94}$$

which, in view of Lin and Cai [18, Table 3], leads to the fact that $\theta = -\pi/2$ is an entrance for interval $[-\pi/2, -\pi/4]$.

Since at $-\pi/4$,

$$\sigma^2(\theta)|_{\theta=-\pi/4} = 0, \quad \mu(\theta)|_{\theta=-\pi/4} = \begin{cases} -8(\delta_1 - \delta_2) < 0, & \delta_1 > \delta_2, \\ -8(\delta_1 - \delta_2) > 0, & \delta_1 < \delta_2, \\ 0, & \delta_1 = \delta_2, \end{cases}$$

$$\alpha_r = 2, \quad \beta_r = \begin{cases} 0, & \delta_1 \neq \delta_2, \\ 1, & \delta_1 = \delta_2. \end{cases} \tag{95}$$

The diffusion behaviors at such a boundary should be discussed for three cases. According to Table 2 in Lin and Cai [18], which gives the classifications of singular boundaries of the first kind, we know that if $\delta_1 > \delta_2$, $-\pi/4$ is an entrance, and if $\delta_1 < \delta_2$, $-\pi/4$ is an exit instead. To determine the boundary type for the case of $\delta_1 = \delta_2$, the character value, which is defined as

$$c_r = -\lim_{\theta \to -\pi^+/2} \frac{-2\mu(\theta)[\theta + (\pi/2)]^{1-0}}{\sigma^2(\theta)} = -\frac{1}{2} \tag{96}$$

is needed. By contrasting Eq. (96) with the relevant terms in Table 2 of Lin and Cai [18], we found that $-\pi/4$ is an attractively natural boundary (ANB). These results are shown in Fig. 2.

Let us consider the interval $[-\pi/4, \pi/4]$. It is easy for us to check the following facts:

If $\delta_1 > \delta_2$, $-\pi/4$ is an exit and $\pi/4$ is an entrance; if $\delta_1 < \delta_2$, instead, $-\pi/4$ is an entrance and $\pi/4$ is an exit. For the case of $\delta_1 = \delta_2$, $-\pi/4$ and $\pi/4$ are both ANB. The results are depicted in Fig. 3.

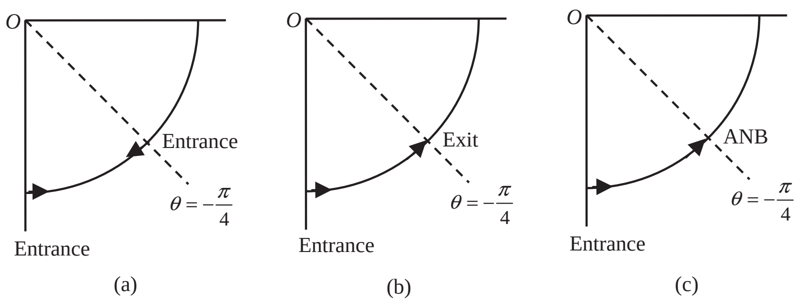


Fig. 2. Boundary diffusion behavior of interval $[-\pi/2, -\pi/4]$ for the cases of (a) $\delta_1 > \delta_2$, (b) $\delta_1 < \delta_2$, (c) $\delta_1 = \delta_2$.

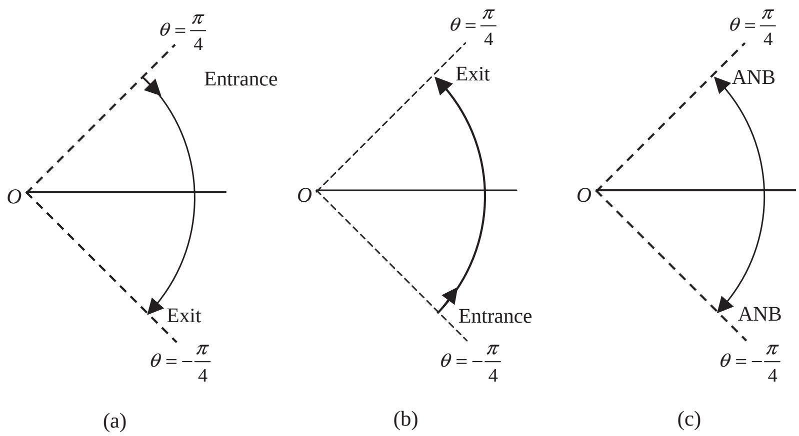


Fig. 3. Boundary diffusion behavior of interval $[-\pi/4, \pi/4]$ for the cases of (a) $\delta_1 > \delta_2$, (b) $\delta_1 < \delta_2$, (c) $\delta_1 = \delta_2$.

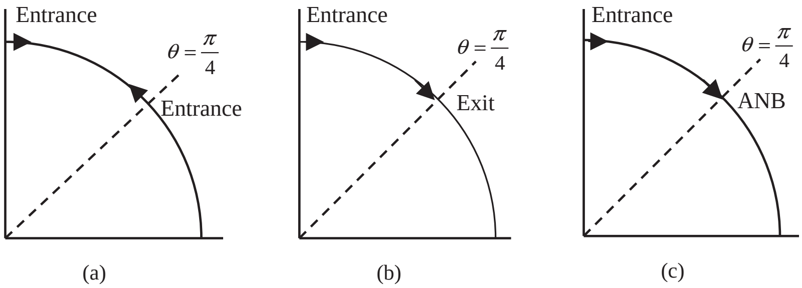


Fig. 4. Boundary diffusion behavior of interval $[\pi/4, \pi/2]$ for the cases of (a) $\delta_1 > \delta_2$, (b) $\delta_1 < \delta_2$, (c) $\delta_1 = \delta_2$.

On $[\pi/4, \pi/2]$, via the same procedure, we know:

If $\delta_1 > \delta_2$, $\pi/4$ is an entrance, if $\delta_1 < \delta_2$, $\pi/4$ is an exit and if $\delta_1 = \delta_2$, $\pi/4$ is an ANB. The other boundary $\pi/2$ is always an entrance on such an interval. The situations are summarized in Fig. 4.

A stationary solution to a FPK equation does not exist [18], if each of the two boundaries is either an exit, or attractively natural, or strictly natural, from which we found that under the conditions of $\delta_1 > \delta_2$ and $\delta_1 = \delta_2$, the invariant measures do not exist on $[-\pi/4, \pi/4]$ and $[-\pi/2, \pi/2]$, respectively, so in this study, the stationary solution to Eq. (66) will be able to be investigated only under the condition of $\delta_1 < \delta_2$.

Next, upon each sub-interval, we determine the stationary solution to Eq. (66). First we consider interval $[-\pi/2, -\pi/4]$. Since $\theta = -\pi/2$ is an entrance and meanwhile $\theta = -\pi/4$ is an exit, we know that on $[-\pi/2, -\pi/4]$, the solution to Eq. (66) is a Direc Delta function which is of the following form [18]:

$$F(\theta) = C\delta\left(\theta + \frac{\pi}{4}\right), \quad \theta \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right], \tag{97}$$

where $C$ is an integral constant which can be determined by the normalization condition of $F(\theta)$ on the whole interval $[-\pi/2, \pi/2]$.

Similarly on interval $[\pi/4, \pi/2]$, the invariant measure is also a Direc Delta function, i.e.,

$$F(\theta) = C\delta\left(\theta - \frac{\pi}{4}\right), \quad \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]. \tag{98}$$

On interval $[-\pi/4, \pi/4]$, since the two boundaries are both entrances, the invariant measure is

$$F(\theta) = Cm(\theta) = \frac{C}{8\kappa_5}[\sec 2\theta]^{3/2} \cos\theta \exp\left[\frac{\delta_1 - \delta_2}{\kappa_5} \sec 2\theta\right], \quad \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]. \tag{99}$$

## 6. Asymptotic expansion for top Lyapunov exponent

Under the assumption that the FPK operator $L_\varepsilon$ defined by Eq. (22) is an ergodic operator, therefore, on the domain $[0, 2\pi] \times [-\pi/2, \pi/2] \times R^n$, the maximal Lyapunov exponent for the stochastic bifurcation system (18) is given as

$$\lambda_\varepsilon = \langle \rho_\varepsilon, p_\varepsilon \rangle = \int_0^{2\pi} \mathrm{d}\phi \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int \mathrm{d}\mathbf{u}[\rho_\varepsilon p_\varepsilon], \tag{100}$$

where $p_\varepsilon(\theta, \phi, \mathbf{u})$ is the stationary probability density which admits the asymptotic expansion (25), and $\rho_\varepsilon$ is defined in Eq. (19). According to the discussion in Section 3, it can be shown easily that the asymptotic expansion of the top Lyapunov exponent

$$\langle \rho_\varepsilon, p_0 \rangle = \varepsilon \langle u_1 \rho_1, p_0 \rangle + \varepsilon^2 [-\langle \rho_2, p_0 \rangle + \langle \rho_1, p_1 \rangle] + \cdots \tag{101}$$

is reasonable. For the stochastic vector process $\mathbf{u}(t)$, since $\langle \mathbf{u}(t) \rangle = 0$, then

$$\langle u_1 \rho_1(\theta, \phi), p_0 \rangle = \frac{1}{2\pi} \int u_1 \psi_0(\mathbf{u}) \mathrm{d}\mathbf{u} \int_0^{2\pi} \mathrm{d}\phi \int_{-\pi/2}^{\pi/2} \rho_1(\theta, \phi) F(\theta) \mathrm{d}\theta = 0. \tag{102}$$

Hence, as a result, the asymptotic expansion of the relevant maximal Lyapunov exponent for system (18) arises as

$$\lambda_\varepsilon = \varepsilon^2 [-\langle \rho_2, p_0 \rangle + \langle u_1 \rho_1, p_1 \rangle] + o(\varepsilon^2). \tag{103}$$

In calculating the asymptotic expansion for the top Lyapunov exponent, the computations of the solution functions as $p_k^{(1)}(\theta, \phi)$, $(k = 1, \ldots, n)$ are needed.

### 6.1. Case 1

For the first case, based on the results of Eq. (92), we obtain

$$p_k^{(1)}(\theta, \phi) = \frac{\gamma_k}{2\pi} \{ \Pi_0^{(k)} \Lambda_0 + \Pi_1^{(k)} \Lambda_1 + \Pi_2^{(k)} \Lambda_2 \}, \tag{104}$$

where

$$\begin{aligned} \Lambda_0 &= \frac{\mathrm{d}}{\mathrm{d}\theta}[F(\theta) \sin 2\theta] \\ &= \cos\theta[-1 + 3\cos 2\theta + \alpha \sin 2\theta] \exp[\alpha(\sin\theta)^2], \end{aligned}$$

$$\Lambda_1 = \tfrac{1}{2}[\cos\theta]^3[3+\alpha-\alpha\cos 2\theta]\exp[\alpha(\sin\theta)^2],$$
$$\Lambda_2 = 2\sin\theta[\cos\theta]^2[-3+\alpha\cos 2\theta]\exp[\alpha(\sin\theta)^2], \tag{105}$$
$$\Pi_0^{(k)} = \frac{1}{2\alpha_k}\left(b_{33}-\frac{k_1}{2}\right),$$
$$\Pi_1^{(k)} = -\frac{1}{(2\omega)^2+\alpha_k^2}\{[\alpha_k k_2+2\omega k_3]\cos 2\phi+[\alpha_k k_3-2\omega k_2]\sin 2\phi\},$$
$$\Pi_1^{(k)} = \frac{1}{(\omega)^2+\alpha_k^2}\{[\alpha_k b_{23}+\omega b_{13}]\cos\phi+[\alpha_k b_{13}-\omega b_{23}]\sin\phi\}.$$

Substitution of Eqs. (104) and (105) into Eq. (103) yields

$$\langle u_1\rho_1, p_1\rangle = \frac{1}{2}\left(\kappa_5+\frac{\beta_2}{4}\right)+\frac{J_2}{2J_0}\left(\kappa_5-\frac{\beta_2}{4}\right),$$
$$\langle \rho_2, p_0\rangle = \delta_1+\frac{J_2}{J_0}(\delta_2-\delta_1),$$
$$J_0 = \int_0^1 [\exp(\alpha x^2)]\,dx, \quad J_2 = \int_0^1 [x^2\exp(\alpha x^2)]\,dx. \tag{106}$$

Then the asymptotic expansion of the top Lyapunov exponent is given as

$$\lambda_\varepsilon = \begin{cases} \varepsilon^2\left[-\delta_1+\dfrac{\kappa_5}{4}+\dfrac{\beta_2}{8}+\dfrac{\kappa_5}{4\sqrt{\pi}}\dfrac{i\exp[\alpha]\sqrt{\alpha}}{Erfi[\sqrt{\alpha}]}\right]+o(\varepsilon^2), & \alpha>0,\\ \varepsilon^2\left[-\delta_1+\dfrac{\kappa_5}{4}+\dfrac{\beta_2}{8}+\dfrac{\kappa_5}{4\sqrt{\pi}}\dfrac{\exp[\alpha]\sqrt{-\alpha}}{Erf[\sqrt{-\alpha}]}\right]+o(\varepsilon^2), & \alpha\leqslant 0. \end{cases} \tag{107}$$

From Eq. (87), we found that condition $\alpha = 0$ means that

$$\delta_1 = \delta_2+[\tfrac{1}{8}S_{u_1}(2\omega)[k_2^2+k_3^2]-\tfrac{1}{2}S_{u_1}(\omega)[b_{13}^2+b_{23}^2]], \tag{108}$$

from which it is easy for us to obtain the inequalities in Eq. (107).

### 6.2. Case 2

For the second case, since

$$M_0 = M_1 = M_2 = 0,$$
$$M_3 = b_{23}\Lambda, \quad M_4 = b_{13}\Lambda, \quad \Lambda = -\mathrm{tg}\,\theta F(\theta)+\frac{d}{d\theta}[\cos 2\theta F(\theta)],$$
$$p_k^{(1)}(\theta,\phi) = \frac{\gamma_k}{2\pi}\frac{1}{\omega^2+\alpha_k^2}\{[\alpha_k M_3+\omega M_4]\cos\phi+[\alpha_k M_4-\omega M_3]\sin\phi\}$$
$$= \frac{\gamma_k}{2\pi}\frac{1}{\omega^2+\alpha_k^2}\{[\alpha_k b_{23}+\omega b_{13}]\cos\phi+[\alpha_k b_{13}-\omega b_{23}]\sin\phi\}\Lambda. \tag{109}$$

The terms in expression (103) can be evaluated as

$$-\langle \rho_2, p_0 \rangle = -\frac{1}{2}(\delta_1+\delta_2)+\frac{1}{2}(\delta_2-\delta_1)\frac{I_1}{I_0}-(\delta_2-\delta_1)\frac{I_1}{I_0}C,$$
$$\langle u_1\rho_1, p_1 \rangle = \frac{1}{4}\kappa_5+2\kappa_5 C+\frac{1}{4}\kappa_5\left[2\frac{I_2}{I_0}-\frac{I_1}{I_0}\right]-\frac{1}{2}\kappa_5 C\left[2\frac{I_2}{I_0}-\frac{I_1}{I_0}\right], \tag{110}$$

where

$$I_0=\int_{-\infty}^{+\infty}\exp[-\kappa x^2]\,\mathrm{d}x=\frac{\sqrt{\pi}}{\sqrt{\kappa}},$$
$$I_1=\int_{-\infty}^{+\infty}\frac{1}{1+x^2}\exp[-\kappa x^2]\,\mathrm{d}x=\pi e^{\kappa}(1-erf(\sqrt{\kappa})),$$
$$I_2=\int_{-\infty}^{+\infty}\left[\frac{1}{1+x^2}\right]^2\exp[-\kappa x^2]\,\mathrm{d}x,$$
$$=\sqrt{\pi\kappa}+\tfrac{1}{2}\pi e^{\kappa}(1-2\kappa)(1-erf(\sqrt{\kappa})), \tag{111}$$
$$C^{-1}=\left[2+\frac{I_0}{8\sqrt{2}\kappa_5 e^{\kappa}}\right],$$
$$\kappa=\frac{\delta_2-\delta_1}{\kappa_5}>0.$$

Then for the second case, the analytical expression of the top Lyapunov exponent of system (18) is obtained as

$$\lambda_\varepsilon=\varepsilon^2\left\{-\delta_1+\frac{\kappa_5}{4}+\lambda_2^{(1)}+\lambda_2^{(2)}\right\}+o(\varepsilon^2),$$
$$\lambda_2^{(1)}=\frac{1}{2}\left\{\left[(\delta_2-\delta_1)-\frac{1}{2}\kappa_5\right]R_1+\kappa_5 R_2\right\},$$
$$\lambda_2^{(2)}=C\left\{(\delta_2-\delta_1)(R_1-1)2\kappa_5+\frac{1}{2}\kappa_5(R_1-2R_2)\right\}, \tag{112}$$

where

$$R_1=\frac{I_1}{I_0}=\sqrt{\pi}\sqrt{\kappa}e^{\kappa}[1-erf(\sqrt{\kappa})],$$
$$R_2=\frac{I_2}{I_0}-\kappa=\frac{1}{2}\sqrt{\pi}\sqrt{\kappa}e^{\kappa}[1-2\kappa][1-erf(\sqrt{\kappa})]. \tag{113}$$

## 7. Conclusion

In this paper, an asymptotic expansion for the maximal Lyapunov exponent of a co-dimension two-bifurcation system driven by a small-intensity real noise process has been constructed. To consider a rather general model, the real noise was assumed to be an output of a linear filter system, viz., a zero-mean stationary Gaussian diffusion process, which satisfies detailed balance condition, and thus stochastic averaging method is not available. The method used in the present study involves the use of (1) the asymptotic analysis given by Arnold et al. [6], and (2) the expansion for the eigenvalue spectrum of Fokker–Planck operator.

In addition, two special cases, one of which where the singularity of the diffusion coefficient arises, are considered and the top Lyapunov exponents are evaluated.

## References

[1] R.Z. Khasminskii, Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems, *Theory of Probability and its Applications* 12 (1967) 144–147.

[2] F. Kozin, S. Prodromou, Necessary and sufficient conditions for almost sure sample stability of linear Ito equations, *SIAM Journal on Applied Mathematics* 21 (1971) 413–424.

[3] R.R. Mitchell, F. Kozin, Sample stability of second order linear differential equations with wide band noise coefficients, *SIAM Journal on Applied Mathematics* 27 (1974) 571–605.

[4] K. Nishiooka, On the stability of two-dimensional linear stochastic systems, *Kodai Mathematical Seminar Report* 27 (1976) 211–230.

[5] S.T. Ariaratnam, W.C. Xie, Lyapunov exponent and rotation number of a two-dimensional nilpotent stochastic system, *Dynamics and Stability of Systems* 5 (1990) 1–9.

[6] L. Arnold, G. Papanicolaou, V. Wihstutz, Asymptotic analysis of the Lyapunov exponents and rotation numbers of the random oscillator and applications, *SIAM Journal on Applied Mathematics* 46 (1986) 427–450.

[7] L. Arnold, *Random Dynamical Systems*, Springer, Berlin, 1998.

[8] S.T. Ariaratnam, W.C. Xie, Lyapunov exponents and stochastic stability of coupled linear systems under real noise excitation, *American Society of Mechanical Engineers Journal of Applied Mechanics* 59 (1992) 664–673.

[9] N.S. Namachchivaya, S. Talwar, Maximal Lyapunov exponent and rotation number for stochastically perturbed co-dimension two bifurcation, *Journal of Sound and Vibration* 169 (1993) 349–372.

[10] N.S. Namachchivaya, H.J. Van Roessel, Maximal Lyapunov exponent and rotation numbers for two coupled oscillators driven by real noise, *Journal of Statistical Physics* 71 (1993) 549–567.

[11] C.W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, 2nd Edition, Springer, Berlin, 1997.

[12] X.B. Liu, K.M. Liew, Lyapunov exponent for two nonlinear systems driven by real noises, *Proceedings of the Royal Society of London, Series A* 458 (2002) 2705–2719.

[13] X.B. Liu, K.M. Liew, The Lyapunov exponent for a codimension two bifurcation system driven by a real noise, *International Journal of Non-Linear Mechanics* 38 (2003) 1495–1511.

[14] P. James, R.A. Scott, Extension of eigenfunction-expansion solutions of a Fokker–Planck equation—II. Second order system, *International Journal of Non-Linear Mechanics* 15 (1980) 41–56.

[15] R.V. Roy, Stochastic averaging of oscillators excited by coloured Gaussian processes, *International Journal of Non-Linear Mechanics* 29 (1994) 461–475.

[16] E. Pardoux, V. Wihstutz, Lyapunov exponent and rotation number of two-dimensional linear stochastic systems with small diffusion, *SIAM Journal on Applied Mathematics* 48 (1988) 442–457.

[17] J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, New York, 1983.

[18] Y.K. Lin, G.Q. Cai, Stochastic stability of nonlinear systems, *International Journal of Non-Linear Mechanics* 29 (1994) 539–553.

[19] S. Karlin, H.M. Taylor, *A Second Course in Stochastic Processes*, Academic Press, New York, 1981.